

Brane-World Motion in Compact Dimensions

Brian Greene^{1,2}, Janna Levin^{1,3}, and Maulik Parikh⁴

¹*Institute for Strings, Cosmology and Astroparticle Physics,
Columbia University, New York, NY 10027, USA*

²*Departments of Physics and Mathematics,
Columbia University, New York, NY 10027, USA*

³*Department of Physics and Astronomy, Barnard College
of Columbia University, New York, NY 10027, USA*

⁴*Department of Physics and Beyond: Center for Fundamental Concepts in Science
Arizona State University, Tempe, AZ 85287, USA*

Abstract

The topology of extra dimensions can break global Lorentz invariance, singling out a globally preferred frame even in flat spacetime. Through experiments that probe global topology, an observer can determine her state of motion with respect to the preferred frame. This scenario is realized if we live on a brane universe moving through a flat space with compact extra dimensions. We identify three experimental effects due to the motion of our universe that one could potentially detect using gravitational probes. One of these relates to the peculiar properties of the twin paradox in multiply-connected spacetimes. Another relies on the fact that the Kaluza-Klein modes of any bulk field are sensitive to boundary conditions. A third concerns the modification to the Newtonian potential on a moving brane. Remarkably, we find that even small extra dimensions are detectable by brane observers if the brane is moving sufficiently fast.

I. INTRODUCTION

In higher-dimensional theories of gravity, brane worlds define a surface on which our Standard Model fields display conventional (3+1)-dimensional physics. In many brane-world models, such as the one proposed by ADD [1], the extra dimensions are compact with a flat metric. Even though Lorentz transformations remain local isometries, the compactness of the extra dimensions violates global Lorentz invariance. In particular, when the extra dimensions are compactified by taking a quotient of \mathbb{R}^n (as in toroidal compactification), the direction of identification picks out a globally preferred frame, despite the Minkowski metric. Consequently it is meaningful to speak of the brane's absolute motion; relativism of motion is lost. Moreover, an observer on a brane universe moving at constant velocity can perform globally sensitive experiments to determine the universe's

velocity through the extra dimension.

In this paper, we present three potentially measurable effects of brane motion: (a) gravitons created at the brane take two different periods of time to travel around the compact dimension; (b) the Kaluza-Klein tower splits into a tower of left-moving and a tower of right-moving states, with different spacings; and (c) the Newtonian potential is modified in such a way as to effectively magnify the size of the extra dimension, L , by the Lorentz factor, γ , with the $4d$ effective gravitational coupling, G_N , now related to the $5d$ coupling, G , through

$$G_N = \frac{G}{\gamma L} . \quad (1)$$

Although possibly difficult to observe, if these effects were detectable, they could deliver measures of the size of the extra dimension and the velocity of our brane universe through the extra dimension.

In §II we review the peculiar features of special relativity on cylindrical Minkowski space, in particular the existence of a globally preferred frame. In §III we realize the topological breaking of Lorentz symmetry in terms of brane worlds embedded in flat compact extra dimensions. We point out several different kinds of effects of brane motion. Time-delayed interactions and graviton return times are described in §III A. We obtain the split in the Kaluza-Klein spectrum in §III B. Low-energy deviations from $1/r^2$ gravity that depend on the boost as well as the size of the circle are presented in §III C. In §IV we discuss our underlying assumptions and indicate some directions for further work.

II. SPECIAL RELATIVITY ON A CYLINDER

The principle of relativity asserts the equivalence of all inertial observers. In special relativity, as in Galilean relativity, there are no preferred observers, and only relative inertial motion has meaning. For special relativity, this fundamental tenet is consistent with the statement that spacetime is an \mathbb{R}^n manifold with the Minkowski metric; Lorentz transformations that take one inertial observer into another leave the Minkowski line element invariant. However, the equivalence of all inertial observers breaks down when spacetime has non-trivial topology. This happens even in flat space. Consider the simple example of a two-dimensional cylindrical spacetime, the product of a circle with the time axis. A cylinder of course is intrinsically flat: the metric in every coordinate patch is precisely the Minkowski metric, the Riemann tensor vanishes, and parallel lines do not meet. Yet, despite being locally identical to ordinary Minkowski space, cylindrical Minkowski space has some unusual, even surprising, properties.

For example, because the worldlines of two inertial observers can intersect multiple times as the observers circumnavigate the multiply-connected dimension, the twin paradox takes on a new and more subtle character [2–6]. On a cylinder neither twin needs to have experienced a period of acceleration in order to reunite with the other twin. Both twins move on geodesics. From the absence of noninertial forces, each twin knows he is inertial and could be tempted to conclude that the laws of special relativity should therefore apply. Each would then think of himself as at rest and the other twin as moving, and therefore as younger. Of course, when they meet they cannot both be right.

The resolution of this unaccelerated version of the twin paradox is that, notwithstanding the Minkowski metric, Lorentz symmetry is broken globally by the topology. This is easy to see on the covering space. The cylinder can be obtained from the \mathbb{R}^n universal covering space through a quotient along a spacelike direction. There is then a preferred frame: observers whose worldlines are orthogonal to the axis of identification are special. The existence of a preferred set of observers seems to violate Lorentz invariance and that is because Lorentz invariance *is* violated, globally though not locally, by the topological identification. And generally it is the case that when we quotient a spacetime by a discrete isometry, we break the isometry group globally.¹ In the case of the cylinder, spacetime is still locally Minkowski space and therefore appears to have Lorentz symmetry. But the identification has glued one coordinate to itself at a fixed time, thereby singling out as special those observers whose spatial coordinate coincides with that along the circle, and rendering all other time slices inequivalent. (Put another way, there is a unique spacelike Killing vector whose integral curves form closed orbits.) What distinguishes the twins then are their speeds with respect to the preferred observer. The twin with the higher speed comes back younger (when the twins have the same speed but opposite velocities, they return with the same age, despite their relative motion). In general, the age difference between the twins can be resolved quantitatively by evaluating the proper times of their worldlines in the covering space [4].

More precisely, consider two-dimensional Minkowski space with topology \mathbb{R}^2 and local line element

$$ds^2 = -dt^2 + dy^2 . \quad (2)$$

We wish to compactify along the y direction so that the topology becomes $S^1 \times \mathbb{R}$. To cover the circle we can choose either a single-valued but discontinuous coordinate or a multi-valued but

¹ Unless the quotient is by an element of the center of the isometry group, as happens in the “elliptic” identification of de Sitter space [7].

continuous coordinate. Choosing the latter, we identify the coordinates via

$$\begin{pmatrix} t \\ y \end{pmatrix} \sim \begin{pmatrix} t \\ y + L \end{pmatrix}, \quad (3)$$

where L is the circumference of the circle. By virtue of having selected the y direction, this identification picks out preferred observers, those whose worldlines are orthogonal to the y -axis. For these observers, space is a circle of circumference L .

The existence of a preferred frame makes it meaningful to speak of absolute speed. Indeed, by performing experiments that probe the global topology of the space, an inertial observer can unambiguously determine whether she is moving. Here is a simple experiment an inertial observer could perform to determine absolute motion: Send a probe around the cylinder. Suppose an inertial observer pair-produces two particles that move in opposite directions along the extra spatial direction. Momentum conservation requires that the two particles have opposite velocities as measured by the observer who produced them. When the preferred observer, O , does this experiment, he finds that particles moving at speed s intercept his worldline again at the same time, L/s .

But consider the same experiment performed by a boosted observer, O' . This observer records a quite different result: the particles return to him at *separate* times since the left-moving probe and the right-moving probe intercept O' 's world line at different events (Fig. 1). Suppose the probes are sent out in opposite directions at speed s' , as measured by O' , who is moving to the right with speed β . Let A be the particle moving to the left of the observer, and let B be the particle moving to the right. Then the amount of preferred time that has elapsed when the probes meet O' 's worldline again are

$$t_A = \frac{L}{s'} \left(\frac{1 - s'\beta}{1 - \beta^2} \right) \quad t_B = \frac{L}{s'} \left(\frac{1 + s'\beta}{1 - \beta^2} \right). \quad (4)$$

If the particles are massless ($s' = 1$), the return times, expressed in terms of O' 's proper time, are

$$t'_A = \frac{L}{\gamma(1 + \beta)} \quad t'_B = \frac{L}{\gamma(1 - \beta)}, \quad (5)$$

where the Lorentz factor, γ , is $(1 - \beta^2)^{-1/2}$.

The preferred observer has $\beta = 0$, so the return periods are the same. In general, any inertial observer can deduce his speed with respect to the preferred frame from the return times. Let $\tau_{\text{short}} = t'_A$ be the shorter time and $\tau_{\text{long}} = t'_B$ the longer time, as measured by O' . Then

$$\beta = \frac{1}{s'} \frac{\tau_{\text{long}} - \tau_{\text{short}}}{\tau_{\text{long}} + \tau_{\text{short}}}, \quad (6)$$

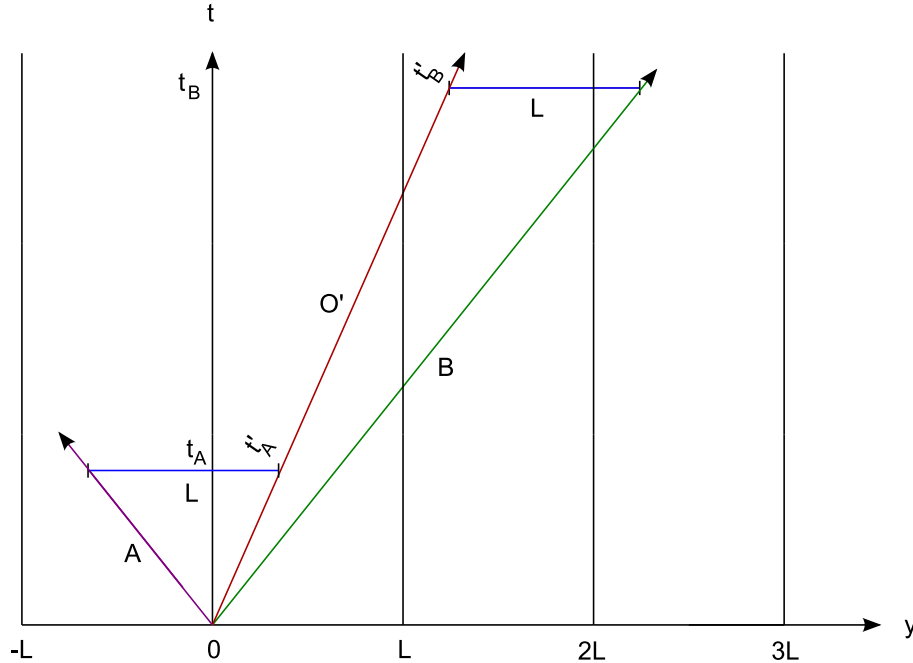


FIG. 1: Spacetime diagram of the universal covering space. The size of the space in the preferred coordinates (t, y) is L . The worldlines of a non-preferred observer, O' , and two particles, A and B , are indicated. Particle A leaves the observer at the origin and returns with period t_A in preferred coordinates and t'_A in the observer's proper time. Particle B returns with a longer period.

where s' is the speed of the probes in the frame of the observer who sent them. In other words, O' can perform the experiment of sending, say, two photons ($s' = 1$) in opposite directions — and upon receipt of those same photons — conclude that he was in uniform motion, without reference to any other observer. In one and only one frame — the preferred frame — do the two returning photons reach the observer simultaneously.² The observer can determine his speed (though not his velocity) with respect to the preferred frame from the difference in photon return times.

A corollary of this is that a family of parallel, moving, inertial observers, at rest with respect to each other, cannot globally synchronize their clocks using Einstein clock synchronization. This is because Einstein clock synchronization methods (sending light signals back and forth) fail to give a unique synchronization for the non-preferred observers: on the cylinder, there is more than one way to send a light signal back, and synchronizing in one direction gives a different result from synchronizing in the other direction. Einstein clock synchronization requires that the travel times of the incoming and outgoing light signals be the same, but this is only true for a pair of preferred

² This resembles the Sagnac effect in which photons traveling down the different arms of a rotating ring-shaped interferometer are subject to different path lengths and transit times, a phenomenon that is important for global positioning system satellites in rotational orbit about the Earth.

observers.

Furthermore, the natural coordinates of moving observers have discontinuities in time as well as in space. Of course, since a circle is not homeomorphic to a segment of the real line, even the preferred observer cannot avoid coordinate discontinuities. But these are just the usual discontinuities in which the spatial coordinate goes from L back to 0. In the moving coordinate system, however, the discontinuities also afflict the time coordinate. Consider an observer O' moving with velocity β relative to the preferred frame. Let t' be his proper time, and let y' coordinatize the spacelike direction orthogonal to $\partial_{t'}$ in the $y-t$ plane. The coordinates (t, y) and (t', y') are related by a Lorentz transformation as follows:

$$\begin{pmatrix} t' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} t \\ y \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}. \quad (7)$$

Acting on both sides of (3) with the Lorentz matrix, Λ , gives the identification in the primed coordinates:

$$\begin{pmatrix} t' \\ y' \end{pmatrix} \sim \begin{pmatrix} t' - \gamma\beta L \\ y' + \gamma L \end{pmatrix}. \quad (8)$$

If a single-valued time coordinate is used, then, at some arbitrary point in space, the time coordinate is forced to jump by a finite amount; evidently, Cartesian coordinates set up by O' suffer discontinuities in both space and time.³

III. MOVING BRANE UNIVERSES

We have seen that the existence of nontrivial topology can break Lorentz invariance globally by selecting a preferred frame. In particular, flat compact extra dimensions provide a concrete realization of this scenario. In the rest of this paper, we will consider brane worlds in flat compact extra dimensions. Since the non-trivial topology has broken global Lorentz invariance and picked

³ Indeed, this is a familiar phenomenon: a similar thing happens across the International Date Line. The world-volume of the surface of the Earth has topology $S^2 \times \mathbb{R}$, of course. But for the purposes of assigning time zones the latitudes play no role; only longitudes matter (neglecting the tilt of the Earth's axis of rotation). So, as far as time zones go, the relevant topology is actually $S^1 \times \mathbb{R}$, a cylinder. (More formally, time zones do not extend to the poles; an S^2 minus the poles is equivalent to an S^1 , by a deformation retraction.) If we were to use preferred time, geostationary clocks everywhere along the Equatorial circle would show the same time. Although such a coordinatization is possible, and even in some sense natural, on Earth we prefer (not because of relativity, but purely out of convenience) to use a different time, one that tracks the motion of the Sun. Rather than assigning the same time to the entire Equator, we choose a Sun-adapted coordinate system that is offset by an hour for every 15 degrees, relative to preferred time. Because such equal-time slices are tilted with respect to the preferred time-slices, they inevitably suffer temporal discontinuities, making an International Date Line unavoidable.

out a preferred frame, it is meaningful to speak of the brane’s velocity. (Brane worlds are invoked here in order to localize the observer within the extra dimensions, thereby ensuring that our special-relativistic considerations — which were framed in terms of classical worldlines — remain valid; had we used the conventional Kaluza-Klein construction, our four-dimensional world would have been smeared uniformly over the extra dimensions, rather than localized within them.) We will find remarkably that — just as an inertial observer can use globally sensitive probes to determine his absolute motion — a brane observer can detect the motion of the brane universe through the extra dimension by using gravitational probes. Our discussion will be purely kinematical and very generic; we will not refer to any specific brane model. The only condition on the brane model is that the brane live in flat compactified space [1, 8]; the Randall-Sundrum model [9] does not apply, not only because the extra dimension is not compact but also because the curved background does not start with Lorentz isometries.

Consider then a generic brane, to which Standard Model fields are confined, living in a spacetime with flat compact extra dimensions. The compact extra dimensions could be either large or small, compared with the inverse of the cut-off scale; these lead to different effects, three of which are discussed in the following subsections.

A. Time-delayed fireworks

Let us assume at first that there is one large extra dimension with the topology of a circle. Here by “large” we mean large compared to the inverse of the cut-off scale. Bulk particles produced on the brane then have characteristic wavelengths that are much smaller than the scale of the extra dimensions; for instance, if the extra dimensions are of millimeter size, and the UV cut-off on the brane is a TeV, then the ratio of the de Broglie wavelength to the size of the extra dimension is about 10^{-16} . Such particles can effectively be described by wavepackets moving on classical trajectories, i.e. on worldlines. Hence when there are large extra dimensions, both the brane observer and the graviton probes can be treated as moving on classical worldlines, exactly as in the previous section.

This leads to the following effect. Imagine that, at the Large Hadron Collider, a collision of Standard Model particles takes place in which massless bulk fields are excited. The bulk particles travel around the extra dimensions on classical trajectories and return to the brane. When they re-enter the accelerator (from the extra dimension), they interact with the brane to produce Standard Model particles. If all the energy is not deposited at once on the brane, the particles go around

additional times, depositing a little more energy on the brane with each collision, in the form of “fireworks” of Standard Model particles [1, 10, 11]. A particle experimentalist in Geneva would therefore observe a sequence of displaced vertices. These interaction vertices would be equally separated in time with a period given by the time taken by the bulk particles to circumnavigate the extra dimension.

However, if the brane happens to be moving through the extra dimension, then, by the logic of the previous section, the vertices would appear with *two* sets of periods. By considering the difference between these two periods, one could, via (6), deduce the speed of our four-dimensional brane as it sails through the extra dimension. More generally, the bulk gravitons could carry off momentum with components tangential to the brane. For a brane with speed β , the following equations are then obeyed in preferred coordinates:

$$\begin{aligned}
 \beta t_{\text{long}} + L &= v_{y,\text{long}} t_{\text{long}} \\
 \beta t_{\text{short}} - L &= -v_{y,\text{short}} t_{\text{short}} \\
 x_{\text{long}} &= v_{x,\text{long}} t_{\text{long}} \\
 x_{\text{short}} &= v_{x,\text{short}} t_{\text{short}} .
 \end{aligned} \tag{9}$$

Here y is the compact direction while x is a direction tangential to the brane. The subscripts long and short label whether the graviton took more or less time respectively to return to the brane. x_{long} and x_{short} are the spatial distances along the brane between the point of creation of the pair of gravitons and the points of return of the gravitons. These equations can be expressed in moving coordinates. The x coordinates are invariant since the boost is transverse to the x -direction, while

$$\begin{aligned}
 t_{\text{long}} &= \gamma \tau_{\text{long}} \\
 t_{\text{short}} &= \gamma \tau_{\text{short}} .
 \end{aligned} \tag{10}$$

The quantities τ_{long} , τ_{short} , x_{long} , and x_{short} are all measurable by an observer on the brane. We therefore have four equations in six unknowns. However, because the gravitons are massless, they also obey

$$\begin{aligned}
 v_{x,\text{long}}^2 + v_{y,\text{long}}^2 &= 1 \\
 v_{x,\text{short}}^2 + v_{y,\text{short}}^2 &= 1 .
 \end{aligned} \tag{11}$$

Hence all six unknowns including the size, L , of the extra dimension, and the speed, β , of the brane can be deduced from the positions in spacetime of graviton interaction vertices. In particular, the

brane speed is

$$\beta = \frac{(\tau_{\text{long}}^2 - \tau_{\text{short}}^2) - (x_{\text{long}}^2 - x_{\text{short}}^2)}{\sqrt{((\tau_{\text{long}} + \tau_{\text{short}})^2 - (x_{\text{long}} - x_{\text{short}})^2)((\tau_{\text{long}} + \tau_{\text{short}})^2 - (x_{\text{long}} + x_{\text{short}})^2)}}. \quad (12)$$

When there is no tangential motion ($x_{\text{long}} = x_{\text{short}} = 0$), this reduces to (6). Remarkably then, using gravitational probes, the inhabitants of the brane can deduce the speed of their universe. Of course if the graviton has a substantial x -component of velocity, the next point of contact with the brane could well have moved outside of the detector altogether, beyond Geneva, or even beyond the solar system. Perhaps another signal of brane motion could be a modification to the missing energy.

We have assumed that particles entering the extra dimension return with probability one. In fact, the returning bulk particles have some interaction probability with the brane. If this probability is not close to unity, then bulk particles may occasionally pass through the brane without producing any interaction vertices. This would then create a set of staggered displaced vertices in which a certain fraction of vertices would be missing, depending on the interaction probability. If there are enough vertices though, one might still be able to determine the shortest period between interactions.

Unfortunately, this effect is likely to be very difficult to measure in practice because of the extreme smallness of the gravitational coupling. Moreover, another complication arises when there is more than one multiply-connected large extra dimension. In that case, point particles leaving the brane will not necessarily return to the brane. Consider a two-torus with modulus $\tau = \tau_1 + i\tau_2$. A point-like particle will return to the brane only if it is moving with slope

$$\frac{n\tau_2}{n\tau_1 + m}, \quad (13)$$

where m and n are integers. But particles moving with such velocities form a set of measure zero.

B. Kaluza-Klein Modes

Now let us consider the opposite limit, in which the size of the extra dimension is comparable to the typical wavelength of the graviton. In this case, the graviton needs to be treated as a wave. Here too there are effects of brane motion, essentially because waves are sensitive to boundary conditions. Consider a free massive scalar field $\phi(t, \vec{x}, y)$. In preferred coordinates, the field obeys the Klein-Gordon equation:

$$(\square - m^2)\phi(t, \vec{x}, y) = (-\partial_t^2 + \vec{\nabla}_x^2 + \partial_y^2 - m^2)\phi(t, \vec{x}, y) = 0. \quad (14)$$

The mode functions are

$$\phi_{k,q} \sim e^{-i\omega t} e^{ikx} e^{iqy} , \quad (15)$$

where

$$\omega^2 = k^2 + q^2 + m^2 . \quad (16)$$

Single-valuedness of the field under $y \sim y + L$ (Eq. (3)) implies that

$$q = \frac{2\pi n}{L} , \quad (17)$$

where n is any integer. This is the usual familiar story.

Now consider the frame moving in the $+y$ direction with speed β . The local line element in the primed coordinates is the usual flat space one and hence the wave operator is also the same. We therefore write

$$(\square' - m^2)\phi(t', \vec{x}', y') = (-\partial_t'^2 + \vec{\nabla}_x'^2 + \partial_y'^2 - m^2)\phi(t', \vec{x}', y') = 0 . \quad (18)$$

For the mode functions this means

$$\phi_{k',q'} \sim e^{-i\omega' t'} e^{ik' x'} e^{iq' y'} , \quad (19)$$

where again

$$\omega'^2 = k'^2 + q'^2 + m^2 . \quad (20)$$

However, in primed coordinates, the identification is given by (8), which mixes space and time components. Single-valuedness of the field under this identification requires that

$$e^{-i\omega' t'} e^{ik' x'} e^{iq' y'} = e^{-i\omega' (t' - \gamma\beta L)} e^{ik' x'} e^{iq' (y' + \gamma L)} . \quad (21)$$

Hence

$$\gamma\beta\omega' + \gamma q' = \frac{2\pi n}{L} = q . \quad (22)$$

This is just the inverse Lorentz transformation acting on the momentum. (There is also the corresponding equation for ω , namely $\gamma\beta q' + \gamma\omega' = \omega$.) We see that, in the non-preferred frame, boundary conditions discretize a linear combination of momentum and energy, in contrast to (17). Substituting for q' into (20), we find that

$$\omega' = \gamma \sqrt{k'^2 + m^2 + \left(\frac{2\pi n}{L}\right)^2} - \beta \frac{2\pi n \gamma}{L} . \quad (23)$$

For the preferred observer, the left- and right-moving modes enter symmetrically in the dispersion relation:

$$\omega^2 = k^2 + m^2 + \left(\frac{2\pi n}{L}\right)^2 \quad (24)$$

In particular, the standard Kaluza-Klein tower has a two-fold degeneracy, with positive and negative n having the same energy; this reflects the fact that left-moving and right-moving bulk modes are treated symmetrically when there is no brane motion. For massless fields with $|k| \ll 2\pi|n|/L$, the energy spectrum has a two-fold degeneracy with fixed spacing $\omega \approx 2\pi|n|/L$. However, when there is brane motion, the degeneracy is lifted. From (23) we have for low $|k|$ that

$$\begin{aligned} \omega' \approx q' &\approx \frac{2\pi n}{\gamma L(1+\beta)} & n > 0 \\ \omega' \approx q' &\approx \frac{2\pi|n|}{\gamma L(1-\beta)} & n < 0 \end{aligned} \quad (25)$$

The Kaluza-Klein tower of states splits up into two interlaced towers. The tower of left-moving states and the tower of right-moving states have different spacings. As $\beta \rightarrow 1$, the frequency of the right-moving states $\rightarrow 0$, and therefore they become easier to excite, while the frequency of the left-moving states $\rightarrow \infty$, and therefore they become harder to excite.

In terms of the corresponding wavelength for massless $k = 0$ modes,

$$\begin{aligned} \lambda_n &= \frac{\gamma L}{n}(1 + \beta) & n > 0 \\ \lambda_n &= \frac{\gamma L}{|n|}(1 - \beta) & n < 0 \end{aligned} \quad (26)$$

We have expressed the factors in this way to suggest that an observer on a moving brane perceives a *larger* extra dimension with length γL . Indeed, (8) already indicates that, in order for the invariant interval between identified points to be unchanged, the identification along the spatial axis is larger in primed coordinates than in preferred coordinates. This magnification of extra dimensions by a Lorentz factor can be visualized as follows. For the moving observer, the spatial axis is tilted with respect to that of the preferred observer. Along this tilted axis, the proper length of the extra dimension once it spirals around and intersects the observer's worldline is indeed γL , as illustrated in Fig 2. In the next section, we will see that the combination γL is also the physically relevant scale in the modification of the Newtonian potential.

The factor of $(1 \pm \beta)$ in (26) is similarly understood. The right-moving standing wave that corresponds to the lowest eigenmode ($n = 1$) can reconnect with its origin on the brane only after it catches up to the brane and so extends over a distance even larger than the brane measures. The left-moving standing wave that corresponds to the lowest eigenmode ($n = -1$) can reconnect with

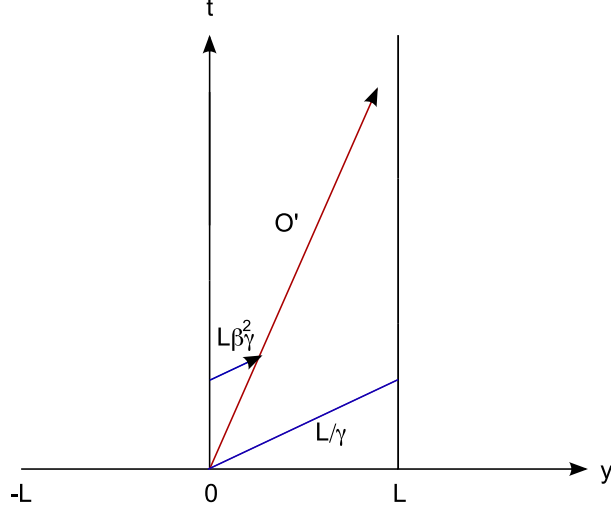


FIG. 2: The effective size of the extra dimension for the moving observer is $L/\gamma + L\beta^2\gamma = \gamma L$.

its origin on the brane when the brane closes the gap to meet it and so extends over a distance smaller than the size of the internal dimension that the brane measures.

C. Newtonian potential on a moving brane

Apart from signatures at accelerators, brane motion also affects the Newtonian potential. Here we calculate the departure of the four-dimensional gravitational potential from the Newtonian $1/r$ form, as seen by an observer on a moving brane. Perhaps surprisingly, the effect is potentially measurable even if the extra dimensions are very small, as they are in standard Kaluza-Klein compactification, provided that the brane is moving sufficiently quickly through the extra dimensions.

We are interested in the Newtonian potential between two sources on the moving brane that are at rest with respect to each other. We calculate the corrected gravitational potential in preferred five dimensional coordinates, $X = (t, \vec{x}, y)$, and transform the result. The locations of mass m_1 and mass m_2 are

$$X_1 = \begin{pmatrix} t_1 \\ \vec{x}_1 \\ y_1 = \beta t_1 \end{pmatrix}, X_2 = \begin{pmatrix} t_2 \\ \vec{x}_2 \\ y_2 = \beta t_2 \end{pmatrix}. \quad (27)$$

The graviton exchange occurs between masses separated in time by $\Delta t = t_1 - t_2$ and on the brane by $\vec{r} = \vec{x}_1 - \vec{x}_2$. As both masses are located on the brane, they are displaced from each other in

the extra direction by $\Delta y = y_1 - y_2 = \beta \Delta t$ according to preferred observers. Hence

$$\Delta X = X_1 - X_2 = \begin{pmatrix} \Delta t \\ \vec{r} \\ \beta \Delta t \end{pmatrix}. \quad (28)$$

From the path integral representation, $Z = e^{iW}$, the interaction potential is computed from

$$W = -\frac{1}{2} \int d^5 X \int d^5 \bar{X} T^{\mu\nu}(X) D_{\mu\nu,\lambda\sigma}(X - \bar{X}) T^{\lambda\sigma}(\bar{X}), \quad (29)$$

where $D_{\mu\nu,\lambda\sigma}$ is the graviton propagator. Since all indices are fully contracted over we can express the resultant scalar in terms of rest mass quantities and hereafter drop the tensor indices on the propagator to treat the graviton as a massless scalar. The contraction is simplest in the frame of the brane where the only non-zero contribution to the energy momentum tensor is

$$T^{t't'} = m_1 \delta^3(\vec{x}' - \vec{x}'_1) \delta(y') + m_2 \delta^3(\vec{x}' - \vec{x}'_2) \delta(y'). \quad (30)$$

Writing

$$W = -\frac{8\pi G}{2} \int d^5 X \int d^5 \bar{X} J(X) D_F(X - \bar{X}) J(\bar{X}), \quad (31)$$

the factor of $8\pi G$ assures the correct Newtonian limit for canonically normalized fields and G is the five-dimensional gravitational constant. The corresponding scalar source is

$$\begin{aligned} J &= m_1 \delta^3(\vec{x}' - \vec{x}'_1) \delta(y') + m_2 \delta^3(\vec{x}' - \vec{x}'_2) \delta(y') \\ &= m_1 \delta^3(\vec{x} - \vec{x}_1) \delta(y - \beta t_1) / \gamma + m_2 \delta^3(\vec{x} - \vec{x}_2) \delta(y - \beta t_2) / \gamma, \end{aligned} \quad (32)$$

where we have used $\delta(az) = \delta(z)/a$. The m_1, m_2 are rest masses and $D_F(X_1 - X_2)$ is the Feynman propagator for the canonically normalized massless scalar field that models the graviton exchange. We integrate over the 8 delta functions and change the remaining integrations over $\int dX^0 \int d\bar{X}^0$ to $\int dT \int d(\Delta t)$. Then, not forgetting a factor of 2 from the $m_1 m_2$ cross terms, the interaction piece in (31) becomes

$$W_{\text{int}}(r) = -T 8\pi G \frac{m_1 m_2}{\gamma^2} \int_{-\infty}^{+\infty} d(\Delta t) D_F(X_1 - X_2). \quad (33)$$

Without identification along y (in our mostly plus metric signature), we would have

$$D_F(\Delta X) = - \int \frac{d^5 k}{(2\pi)^5} \frac{e^{ik \cdot \Delta X}}{k^2 - i\epsilon}. \quad (34)$$

But because y is compact, k_y is discretized in units of $2\pi/L$ so that $dk_y/(2\pi) = 1/L$ and the fifth integral is replaced by a sum. The Newtonian potential is therefore

$$W_{\text{int}}(r) = T \frac{8\pi G m_1 m_2}{(2\pi)^4 \gamma^2} \sum_{n=-\infty}^{n=+\infty} \frac{1}{L} \int \frac{e^{ikr \cos \theta_k} k^2 \sin \theta_k}{-\omega^2 + k^2 + \left(\frac{2\pi n}{L}\right)^2 - i\epsilon} dk d\theta_k d\phi_k d\omega \int d(\Delta t) e^{-i(\omega - 2\pi n \beta/L)\Delta t} . \quad (35)$$

The Δt integral gives $2\pi\delta(\omega - 2\pi n \beta/L)$. Since the ω integral runs from 0 to ∞ , the delta function eliminates the negative n part of the sum, and integrates to 1/2 when $n = 0$. After the angular integrations, we obtain

$$W_{\text{int}}(r) = -iT \frac{8\pi G m_1 m_2}{(2\pi)^2 \gamma^2 L r} \left(\frac{1}{2} \int_{-\infty}^{+\infty} \frac{k e^{ikr}}{k^2 - i\epsilon} dk + \sum_{n=+1}^{n=+\infty} \int_{-\infty}^{+\infty} \frac{k e^{ikr}}{k^2 + \left(\frac{2\pi n}{\gamma L}\right)^2 - i\epsilon} dk \right) . \quad (36)$$

The integrals over k can be written as closed contours in the upper half-plane giving

$$W_{\text{int}}(r) = T \frac{2G m_1 m_2}{\gamma^2 L r} \left(\frac{1}{2} + \sum_{n=+1}^{n=+\infty} e^{-2\pi n r / (\gamma L)} \right) . \quad (37)$$

so that

$$W_{\text{int}}(r) = T \frac{G m_1 m_2}{\gamma^2 L r} \frac{1 + e^{-2\pi r / (\gamma L)}}{1 - e^{-2\pi r / (\gamma L)}} . \quad (38)$$

Now W is a scalar so $Z = e^{iW}$ is true in any frame. In brane coordinates the particles are at rest and the energy is pure potential, $Z = e^{-iH'T'} = e^{-iV_{\text{brane}}(r)T'}$. Since $T/T' = \gamma$ we have finally the modified Newtonian potential as measured by observers at rest on the brane:

$$V_{\text{brane}}(r) = -\frac{G m_1 m_2}{\gamma L r} \frac{1 + e^{-2\pi r / (\gamma L)}}{1 - e^{-2\pi r / (\gamma L)}} . \quad (39)$$

When $r \ll \gamma L$, $V_{\text{brane}}(r)$ behaves like $-\frac{G m_1 m_2}{\pi r^2}$, which is indeed the five-dimensional Newtonian potential. On the other hand, at large distances, $r \gg \gamma L$, $V_{\text{brane}}(r)$ goes as $-\frac{G m_1 m_2}{\gamma L r}$, which is just the four-dimensional potential, provided we define the effective four-dimensional Newton's constant, G_N , to be

$$G_N = \frac{G}{\gamma L} . \quad (40)$$

Remarkably, in (39) and (40) the effective size of the compact space is γL , rather than L . The extra dimensions appear *magnified*. Corrections to the $1/r$ form of the gravitational potential arise when r becomes appreciable compared to γL , rather than L .

This is a very interesting result because it means that even if the extra dimensions are very small, as in standard Kaluza-Klein compactification on a torus, we could still detect them as

deviations from the four-dimensional Newtonian potential if, for some reason, our brane were moving at an ultra-relativistic speed through the extra dimensions. Through precision table-top gravity experiments, one might be able to extract the value of γL from data. In that case, the other effects described earlier (difference in graviton return times, splitting of the Kaluza-Klein tower) would constitute nontrivial consistency checks between the measurements.

As a check, we can re-derive the result by doing the calculation directly on the covering space. We will use the method of images. Consider ordinary, infinite five-dimensional Minkowski space covered by two coordinate systems, (t, \vec{x}, y) and (t', \vec{x}', y') . On the covering space, neither of these coordinates are in any way pathological. Let the brane be aligned along $y' = 0$ with an infinite number of images displaced by L along the *unprimed* y -axis. This is the covering space picture of our scenario.

On the covering space, the five-dimensional graviton propagator $D_F(X)$ (dropping tensor indices again) is just

$$D_F(X') = \frac{i}{8\pi^2 (X'^2)^{3/2}} , \quad (41)$$

where $X'^2 = -t'^2 + r'^2 + y'^2$ is the five-dimensional invariant distance-squared (see e.g. [12]). Then the Newtonian potential on the brane is

$$V_{\text{brane}}(r) = 4\pi G m_1 m_2 \int_{-\infty}^{+\infty} d(\Delta t') D(\Delta X') . \quad (42)$$

Here D is related to D_F by a sum over images

$$D(\Delta X') = \sum_{n=-\infty}^{n=+\infty} D_F(\Delta X'_n) , \quad (43)$$

where $\Delta X'_n$ is the difference between the source at the origin and the images of the second mass. Since the identification in the primed coordinates is

$$\begin{pmatrix} t' \\ \vec{x}' \\ y' \end{pmatrix} \sim \begin{pmatrix} t' - \gamma\beta nL \\ \vec{x}' \\ y' + \gamma nL \end{pmatrix} , \quad (44)$$

the images of $(t', \vec{r}', y' = 0)$ are separated from the origin by

$$\Delta X'_n = (\Delta t' - \gamma\beta nL, \vec{r}', \gamma nL) . \quad (45)$$

We can now readily evaluate the potential:

$$\begin{aligned}
V_{\text{brane}}(r) &= \frac{Gm_1m_2}{2\pi} \sum_{n=-\infty}^{n=+\infty} \int_{-\infty}^{\infty} \frac{id(\Delta t')}{(-(\Delta t' - \gamma\beta nL)^2 + r^2 + (\gamma nL)^2)^{3/2}} \\
&= -\frac{Gm_1m_2}{\pi} \sum_{n=-\infty}^{n=+\infty} \frac{1}{r^2 + (\gamma nL)^2} \\
&= -\frac{Gm_1m_2}{\gamma L r} \frac{1 + e^{-2\pi r/(\gamma L)}}{1 - e^{-2\pi r/(\gamma L)}}.
\end{aligned} \tag{46}$$

To obtain the second equality we have Wick-rotated ($\tau = it$) the line of integration, and for the last line we used the identity

$$\sum_{n=-\infty}^{n=+\infty} \frac{1}{x^2 + \left(\frac{n\pi}{a}\right)^2} = \frac{a}{x} \coth(ax). \tag{47}$$

Note that, in (46), the combination that appears is again γL . This confirms our earlier result.

IV. DISCUSSION

We have seen that quotient spaces automatically break global Lorentz invariance even when spacetime is precisely flat everywhere. The key point is that the direction of identification picks out a preferred frame and thereby makes it meaningful — despite the Minkowski metric — to speak of the absolute velocity of objects. Individual inertial observers can then determine their state of motion by means of experiments that probe the global topology. In this paper we have pointed out that the exotic special-relativistic effects that arise from the global breaking of Lorentz invariance have a concrete realization in terms of brane worlds moving through flat compact extra dimensions. Brane observers can potentially detect the motion of the brane by experiments involving bulk fields, notably gravity. We found three effects of brane motion: for large extra dimensions, gravitons return to the brane with two sets of periodicities; for small extra dimensions, there is a splitting of the Kaluza-Klein spectrum; and in both cases there is an enhancement of the deviation from the $1/r$ form of the Newtonian potential, which magnifies the extra dimensions by γ . There could well be other interesting effects. Among the several questions this study raises is the naturalness, or unnaturalness, of significant brane velocity — of significant γ . Perhaps brane gas cosmology establishes a velocity distribution of branes that determines if a typical brane would move at relativistic speeds.

We have made one simplifying assumption: we neglected the gravitational backreaction of the brane on the background geometry. This assumption puts a constraint on the brane velocity and

brane tension, which we can estimate heuristically. Consider a 3-brane living in a five-dimensional spacetime. From the dimension-independent Poisson equation, $\nabla^2 V = 4\pi G\rho$, we can integrate over a spatial volume to get Gauss's Law, $-\int f \cdot dA = 4\pi G m_{\text{enc}}$ relating the force per unit test mass to the enclosed source mass. Consider a four-dimensional cube enclosing a piece of the brane. The force per unit test mass goes as $f \propto GT_0$, where T_0 is the brane tension. The gravitational potential due to the brane therefore scales like $V \propto GT_0 y$ as a function of distance y in the extra transverse direction. For the backreaction to be small, we require $V \ll 1$. Now, the boosted energy density is γT_0 . Hence, with $G \propto \ell_P^3$, we find that the requirement that backreaction be small yields a condition on the product γT_0 of the 3-brane

$$G\gamma T_0 L \ll 1 \Rightarrow \gamma \ll \frac{l_P}{L} \frac{1}{T_0 l_P^4} . \quad (48)$$

Analogous considerations for n extra dimensions ($n \geq 3$) indicate that backreaction can be neglected when

$$G\gamma T_0 \left(\frac{1}{L}\right)^{n-2} \ll 1 \Rightarrow \gamma \ll \left(\frac{L}{l_P}\right)^{n-2} \frac{1}{T_0 l_P^4} . \quad (49)$$

If the brane tension is much lower than the Planck scale, the Lorentz factor can be enormous.

Treated more formally, the presence of branes with tension poses difficulties in a compact space. From the higher-dimensional point of view, a brane is a delta-function source for the gravitational field. In a compact space, however, there can be no net mass, just as there can be no net Noether charge. (Gauss's Law gives inconsistent results: a positive source enclosed inside a cube versus zero source enclosed outside a cube.) Thus the mass would have to be canceled somehow. One could cancel it by adding a negative tension brane, such as an orientifold. However, the presence of a second brane would interfere with the ability of gravitons to go around the extra dimension, if there were only one extra dimension. A different possibility would be to consider only a single brane whose mass is canceled by a neutralizing background [13]. It would be interesting to find an explicit solution with brane motion through such a background, in which the background is still sufficiently close to flat so that Lorentz symmetry is still an approximate isometry. These are technical difficulties, not physically prohibitive obstacles. General relativity should allow for brane motion, even if the metric is resistant to analytic solutions. Still, genuine physical obstacles could well interfere, as a time-independent internal space might not be consistent with brane motion for instance. It would be interesting to explore this, for example by considering cosmology on a moving brane. It would also be interesting to consider the effects of brane motion in the context of specific phenomenologically viable brane world models.

Acknowledgments

We would like to thank Gia Dvali, Pedro Ferreira, Lam Hui, Justin Khoury, David Langlois, Edouardo Ponton, Massimo Porrati, Yuri Shtanov, and Kishan Yerubandi for helpful discussions. JL gratefully acknowledges support of NSF Theoretical Physics grant, PHY - 0758022, and a KITP Scholarship.

-
- [1] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, “The hierarchy problem and new dimensions at a millimeter,” *Phys. Lett. B* **429** (1998) 263; [hep-ph/9803315](#).
 - [2] C. H. Brans and D. R. Stewart, “Unaccelerated-returning-twin paradox in flat space-time,” *Phys. Rev. D* **8** (1973) 1662.
 - [3] P. C. Peters, “Periodic boundary conditions in special relativity,” *Am. J. Phys.* **51** (1983) 791.
 - [4] J. D. Barrow and J. J. Levin, “The twin paradox in compact spaces,” *Phys. Rev. A* **63** (2001) 044104; [gr-qc/0101014](#).
 - [5] J. D. Barrow and J. Levin, “The Copernican principle in compact space-times,” *Mon. Not. Roy. Astron. Soc.* **346** (2003) 615; [gr-qc/0304038](#).
 - [6] J.-P. Uzan, J.-P. Luminet, R. Lehoucq, and P. Peter, “Twin paradox and space topology,” *Eur. J. Phys.* **23** (2002) 277; [physics/0006039](#).
 - [7] M. Parikh, I. Savonije, and E. Verlinde, “ dS/\mathbb{Z}_2 : Elliptic de Sitter space,” *Phys. Rev. D* **67** (2003) 064005; [hep-th/0209120](#).
 - [8] G. R. Dvali, G. Gabadadze, and M. Porrati, “4D gravity on a brane in 5D Minkowski space,” *Phys. Lett. B* **485** (2000) 208; [hep-th/0005016](#).
 - [9] L. Randall and R. Sundrum, “An alternative to compactification,” *Phys. Rev. Lett.* **83** (1999) 4690; [hep-th/9906064](#).
 - [10] D. Langlois and L. Sorbo, “Bulk gravitons from a cosmological brane,” *Phys. Rev. D* **68** (2003) 084006; [hep-th/0306281](#).
 - [11] D. Langlois, L. Sorbo, and M. Rodriguez-Martinez, “Cosmology of a brane radiating gravitons into the extra dimension,” *Phys. Rev. Lett.* **89** (2002) 171301; [hep-th/0206146](#).
 - [12] A. O. Barvinsky and S. N. Solodukhin, “Echoing the extra dimension,” *Nucl. Phys.* **B675** (2003) 159; [hep-th/0307011](#).
 - [13] Y. V. Shtanov, “Flat vacuum branes without fine tuning,” *Phys. Lett. B* **543** (2002) 121; [hep-ph/0108211](#).